

Counting function fluctuations and extreme value threshold in multifractal patterns: the case study of an ideal $1/f$ noise

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Abstract. Motivated by the general problem of studying sample-to-sample fluctuations in disorder-generated multifractal patterns we attempt to investigate analytically as well as numerically the statistics of high values of the simplest model - the ideal periodic $1/f$ Gaussian noise. Our main object of interest is the number of points $\mathcal{N}_M(x)$ above a level $\frac{x}{2}V_m$, with $V_m = 2 \ln M$ standing for the leading-order typical value of the absolute maximum for the sample of M points. By employing the thermodynamic formalism we predict the characteristic scale and the precise scaling form of the distribution of $\mathcal{N}_M(x)$ for $0 < x < 2$. We demonstrate that the powerlaw forward tail of the probability density, with exponent controlled by the level x , results in an important difference between the mean and the typical values of $\mathcal{N}_M(x)$. This can be further used to determine the typical threshold x_m of extreme values in the pattern which turns out to be given by $x_m^{(typ)} = 2 - c \ln \ln M / \ln M$ with $c = \frac{3}{2}$. Such observation provides a rather compelling explanation of the mechanism behind universality of c . Revealed mechanisms are conjectured to retain their qualitative validity for a broad class of disorder-generated multifractal fields. In particular, we predict that the typical value of the maximum p_{max} of intensity is to be given by $-\ln p_{max} = \alpha_- \ln M + \frac{3}{2f'(\alpha_-)} \ln \ln M + O(1)$, where $f(\alpha)$ is the corresponding singularity spectrum vanishing at $\alpha = \alpha_- > 0$. For the $1/f$ noise case we further study asymptotic values of the prefactors in scaling laws for the moments of the counting function. Our numerics shows however that one needs prohibitively large sample sizes to reach such asymptotics even with a moderate precision. This motivates us to derive exact as well as well-controlled approximate formulas for the mean and the variance of the counting function without recourse to the thermodynamic formalism.

1. Introduction

Investigations of multifractal structures of diverse origin is for several decades a very active field of research in various branches of applied mathematical sciences like chaos theory, geophysics and oceanology [1, 2] as well as climate studies [3], mathematical finance [4, 5], and in such areas of physics as turbulence [6, 7], growth processes [8], and theory of quantum disordered systems [9]. The main characteristics of a multifractal pattern of data is to possess high variability over a wide range of space or time scales, associated with huge fluctuations in intensity which can be visually detected. Another common feature is presence of certain long-ranged powerlaw-type correlations in data values[10].

To set the notations, consider a certain (e.g. hypercubic) lattice of linear extent L and lattice spacing a in d -dimensional space, with $M \sim (L/a)^d \gg 1$ standing for the total number of sites in the lattice. The multifractal patterns are then usually associated with a set of non-negative "heights" $h_i \geq 0$ attributed to every lattice site $i = 1, 2, \dots, M$ such that the heights scale in the limit $M \rightarrow \infty$ differently at different sites: $h_i \sim M^{x_i}$ §, with exponents x_i forming a dense set. To characterize such a pattern of heights quantitatively it is natural to count the sites with the same scaling behaviour. Then a multifractal measure is characterized by a (usually, concave) single-maximum *singularity spectrum* function $f(x)$. Denoting the position of its maximum as $x = x_0$, such function describes the (large-deviation) scaling of the number of points in the pattern whose local exponents x_i belong to some interval around x_0 . More precisely, defining the density of exponents by $\rho_M(x) = \sum_{i=1}^M \delta\left(\frac{\ln h_i}{\ln M} - x\right)$ a nontrivial multifractality implies that such density should behave in the large- M limit as [11]

$$\rho_M(x) \approx c_M(x) \sqrt{\ln M} M^{f(x)} \quad (1)$$

with a prefactor $c_M(x)$ of the order of unity which may still depend on x . We will refer below to the above form as the *multifractal ansatz*. The major effort in the last decades was directed towards determining the shape and properties of $f(x)$. In contrast, our main object of interest will be the behaviour of the prefactor $c_M(x)$ which is much less studied, to the best of our knowledge. In particular, if the multifractal pattern is randomly generated like e.g. those considered in [9] the ansatz (1) is expected to be valid in every realization of the disorder. One may then be interested in understanding the sample-to-sample fluctuations of the prefactor $c_M(x)$.

§ Usually one defines exponents α_i via the relation $h_i \sim L^{\alpha_i}$ i.e. by the reference to linear scale L instead of the total number of sites $M \sim (L/a)^d$, and similarly for the density of exponents $\rho(\alpha) \sim L^{f(\alpha)}$. We however find it more convenient to use instead the exponents $x_i = \alpha_i/d$ and the singularity spectrum $f(x) = \frac{1}{d}f(\alpha)$.

Further introducing the counting functions

$$N_{>}(x) = \int_x^\infty \rho_M(y) dy, \quad N_{<}(x) = \int_{-\infty}^x \rho_M(y) dy \quad (2)$$

for the total number $N_{>}(x)$ of sites of the lattice where heights satisfy $h_i > M^x$ (respectively, $h_i < M^x$), substituting the multifractal form of the density into (2) and performing at $\ln M \gg 1$ the resulting integral for $x > x_0$ by the Laplace method we find $N_{>}(x) \approx c_M(x) M^{f(x)} / |f'(x)| \sqrt{\ln M}$ and a similar expression for $N_{<}(x)$ for $x < x_0$, relating the singularity spectrum $f(x)$ to the counting functions. As both $N_{>}(x)$ and $N_{<}(x)$ can not be smaller than unity we necessarily have $f(x) \geq 0$ for all x , and the condition $f(x) = 0$ defines generically the maximal x_+ and the minimal x_- threshold values of the exponents which can be observed in a given height pattern.

The singularity spectrum $f(x)$ is not a quantity which is easily calculated analytically or even numerically for a given multifractal pattern of heights[11]. An alternative procedure of analysing the multifractality is frequently referred to in the literature as the *thermodynamic formalism* [1, 2]. In that approach one characterizes the multifractal pattern by the set of exponents ζ_q describing the large- M scaling behaviour of the so-called *partition functions* Z_q as

$$Z_q = \sum_{i=1}^M h_i^q \sim M^{\zeta_q}, \quad \ln M \gg 1 \quad (3)$$

To relate ζ_q to the singularity spectrum $f(x)$ discussed above one rewrites (3) in terms of the density as $Z_q = \int_{-\infty}^\infty M^{qx} \rho_M(x) dx$, and again employs the multifractal ansatz (1) for $\rho_M(x)$. Evaluating the integral in the $\ln M \gg 1$ limit by the steepest descent (Laplace) method gives

$$Z_q \sim c_M(x_*) \left(\frac{2\pi}{|f''(x_*)|} \right)^{1/2} M^{\zeta_q} \quad \text{where} \quad f'(x_*) = -q \quad \text{and} \quad \zeta_q = f(x_*) + q x_*, \quad (4)$$

where we have assumed that $x_- < x_* < x_+$ for simplicity. This shows that the relation between ζ_q and $f(x)$ is given essentially by the *Legendre transform*. We thus see that formally the original definition of multifractality based on the density (or, equivalently, the counting functions $N_{>,<}(x)$) and the thermodynamic formalism approach (3)-(4) should have exactly the same content for $\ln M \rightarrow \infty$, provided the singularity spectrum is concave^{||}. Note also the normalization identity $Z_0 = \int_{-\infty}^\infty \rho_M(y) dy \equiv M$ implying $\zeta_0 = 1$. It also shows that at the point of maximum $x = x_0$ we must necessarily have $f(x_0) = 1$ and that $c_M(x_0)$ is indeed of the order of unity.

In practice, to extract singularity spectra from a given multifractal pattern obtained in real or computer experiments, one frequently employs the so-called *box*

^{||} Examples of non-concave multifractality spectrum and the associated thermodynamic formalism are discussed in [12]

counting procedure which can be briefly described as follows. Subdivide the sample into $M_l = (L/l)^d$ non-overlapping hypercubic boxes Ω_k of linear dimension l . Associate with each box the mean height $H_k(l) = (l/a)^{-d} \sum_{i \in \Omega_k} h_i$, and define the scale-dependent partition functions

$$Z_q(l, L) = \frac{1}{M_l} \sum_{k=1}^{M_l} [H_k(l)]^q \quad (5)$$

Note that for $l = a$ obviously $M_a = M = (L/a)^d$ and $Z_q(a, L)$ coincides with the partition function Z_q featuring in the thermodynamic formalism. One may however observe that in the range $a \ll l \ll L$ the scale-dependent partition functions are sensitive to the spatial correlations in the heights at different lattice points. In particular, a simple consideration shows that when the heights are powerlaw-correlated in space as is actually the case for the most of systems of interest[10] (see also below) the scaling behaviour of $Z_q(l, L)$ depends non-trivially on both l/a and L/a . At the same time the behaviour of the combination $I_q(l, L) = Z_q(l, L) / [Z_1(l, L)]^q$ turns out to be a function only on the scaling ratio L/l and is given by

$$I_q(l, L) = \frac{Z_q(l, L)}{[Z_1(l, L)]^q} \sim \left(\frac{L}{l} \right)^{-\tau_q}, \quad \text{where} \quad \tau_q = d(q\zeta_1 - \zeta_q) \quad (6)$$

which allows to get reliable numerical values of the scaling exponents τ_q by varying the ratio L/l over a big range. Further noticing that for $q = 1$ the l -dependence of the partition function disappears due to linearity: $Z_1(l, L) = (l/a)^{-d} (L/l)^{-d} \sum_1^M h_i \sim (L/a)^{\zeta_1 - d}$ we also can reliably extract ζ_1 from the same data, hence relate the set of exponents τ_q to ζ_q for $q \neq 1$.

The quantities $I_q = I_q(a, L)$ have interpretation of the *inverse participation ratio's* (IPR's) and are very popular in the theory of the Anderson localization [9] and related studies. Passing from the partition functions Z_q of the thermodynamic formalism to the IPR's is equivalent to focusing on the properties of the normalized *probability measure* $0 < p_i = h_i/Z_1 < 1$, $\sum_i p_i = 1$ rather than on the original height pattern h_i itself. In fact in such a setting it is more natural to introduce the scaling of those weights in the form $p_i \sim M^{-\alpha_i}$, $\alpha_i \geq 0$ and consider the corresponding singularity spectrum $f(\alpha)$ related directly to the Legendre transform of the exponents τ_q . Working with the exponents τ_q has some advantages, as one can show they must be monotonically increasing convex function of q : $\frac{d\tau_q}{dq} > 0$, $\frac{d^2\tau_q}{dq^2} \leq 0$. In many situations, as e.g. the diffusion-limited aggregation[8] or indeed the Anderson localization the multifractal probability measures arise very naturally. In other contexts, e.g. in turbulence or in financial data analysis, the normalization condition seems superfluous. In the main part of the present paper we are mainly interested in the pattern of heights and are therefore concentrating on partition functions. We will discuss the normalized multifractal probability measures in the end.

The above description is generally valid for a given single multifractal pattern of any nature, not necessarily random. In recent years considerable efforts were directed towards understanding disorder-generated multifractality, see e.g. [9, 13, 14, 15] for a comprehensive discussion in the context of Anderson localisation transitions and various associated random matrix models, [16], [17] in the context of harmonic measure generated by conformally invariant two-dimensional random curves and [18, 19, 20] for examples related to Statistical Mechanics in disordered media. We just briefly mention here that one of the specific features of multifractality in the presence of disorder is a possibility of existence of two sets of exponents, τ_q versus $\tilde{\tau}_q$, governing the scaling behaviour of the typical IPR denoted $I_q^{(t)} \sim M^{-\tau_q}$ versus disorder averaged ("annealed") IPR, $\overline{I_q} \sim M^{-\tilde{\tau}_q}$. Here and henceforth the overline stands for the averaging over different realisations of the disorder. Namely, it was found that for large enough $q > q_c$ the two exponents will have in general different values: $\tau_q \neq \tilde{\tau}_q$. The possibility of "annealed" average to produce results different from typical is related to a possibility of disorder-averaged moments to be dominated by exponentially rare configurations. As a result, the part of the "annealed" multifractality spectrum recovered via the Legendre transform from $\tilde{\tau}_q$ for $q > q_c$ will be negative [21],[9]: $\tilde{f}(x) < 0$ for $x < x_-$, and similarly for $x > x_+$. Further detail can be found in the cited papers and in the lectures [22].

Another important aspect of random multifractals revealed originally by Mirin and Evers [14] in the context of the Anderson localization transition is the fact that IPR's I_q for disorder-induced multifractal probability measures are generically power-law distributed: $\mathcal{P}(I_q/I_q^{(t)}) \sim (I_q/I_q^{(t)})^{-1-\omega_q}$ [14, 9]. Such behaviour suggests that the actual values of the counting functions $N_>(x), N_<(x)$ should also show substantial sample-to-sample fluctuations, even in the range $x_- < x < x_+$ where the singularity spectrum $f(x)$ is self-averaging and the same multifractal scalings $N_>(x) \sim M^{f(x)}$ is to be observed in every realization of the pattern. Though the presence of such fluctuations was already mentioned in [18], a detailed quantitative analysis seems to be not available yet. The main goal of our paper is to achieve a better understanding of statistics of the counting functions $N_>(x), N_<(x)$ by performing a detailed analytical as well as numerical study of arguably the simplest, yet important class of multifractal disordered patterns - those generated by one-dimensional Gaussian processes with logarithmic correlations, the so-called $1/f$ noises.

The structure of the paper is as follows. In the next section we will introduce the $1/f$ noise signals and discuss their properties already known from the previous works. Then we will use that knowledge to show that the probability density of the counting function $N_>(x)$ for such a model is characterized by a limiting scaling law with a powerlaw forwards tail, with the power governing the decay changing with the level x . We will then demonstrate that such powerlaw decay has nontrivial implications for the position of the maxima (or, with due modifications, minima) of such processes, and

derive the expression for the threshold of extreme values. Finally, using $1/f$ noises as guiding example we will attempt to reinterpret the results of the theory developed in [14] to get a rather general prediction for the position of extreme value threshold for a broad class of disorder-generated multifractal patterns whose intensity is characterized by power-law correlations. We conclude with briefly discussing a few open questions.

2. $1/f$ noise: mathematical model and previous results

An ideal $1/f$ (or "pink") noise is a random signal such that spectral power (defined via the Fourier transform of the autocorrelation function of the signal) associated with a given Fourier harmonic is inversely proportional to the frequency $\omega = 2\pi f$. Signals of similar sort are known for about eighty years and believed to be ubiquitous in Nature, see [23] for a discussion and further references. Rather accurate $1/f$ dependences may extend for several decades in frequency in some instances, es. e.g. in voltage fluctuations in thin-film resistors [24] or resistance fluctuations in single-layer graphene films [25], in non-equilibrium phase transitions [26], and in spontaneous brain activity [27]. Still, the physical mechanisms behind such a behaviour are not yet fully known, and are a matter of active research and debate. It was noticed quite long ago (see e.g. [28]) that a generic feature of all such signals is that the two-point correlations (covariances) depend *logarithmically* on the time separation. During the last decade it became clear that random functions of such type appear in many interesting problems of quite different nature, featuring in physics of disordered systems [18],[29],[30],[31],[32], quantum chaos [33], mathematical finance [34, 35], turbulence [36, 37, 38] and related models [39, 40, 41], as well as in mathematical studies of random conformal curves [42], Gaussian Free Field [43, 44] and related models inspired by applications in statistical mechanics [45] and quantum gravity [46], and the most recently in the value distribution of the characteristic polynomials of random matrices and the Riemann zeta-function along the critical axis [47]. Let us note that a simple argument outlined in [22] and repeated in section 5 of the present paper shows that by taking the logarithm of any spatially homogeneous powerlaw-correlated multifractal random field we necessarily obtain a field logarithmically correlated in space. A somewhat similar in spirit suggestion to refocus the attention from the multifractal ("intermittent") signals and fields to their logarithms was also put forward in [37]. All this makes logarithmically-correlated Gaussian processes an ideal laboratory for studying disorder-induced multifractality, though investigating the effects of non-Gaussianity remains a challenging outstanding issue.

Despite the fact of being of intrinsic interest and fundamental importance, a coherent and comprehensive description of statistical characteristics of ideal $1/f$ noises seems not to be yet available, and relatively few properties are firmly established even for

the simplest case of a Gaussian $1/f$ noise. Among the works which deserve mentioning in such a context is the paper [48] which provided an explicit distribution of the "width" (or "roughness") for such a signal, as well as the work [49] describing a curious property of spectral invariance with respect to amplitude truncation. In recent papers [30],[31] and [45] the statistics of the *extreme* (minimal or maximal) values of various versions of the ideal $1/f$ signals was thoroughly addressed. From that angle the subject of the present paper is to provide a fairly detailed picture of statistics of the number of points in such signals which lie above a given threshold set at some *high* value. The latter can be rather naturally defined as being at finite ratio to the typical value of the absolute maximum.

In this paper we are going to consider only Gaussian ideal $1/f$ noises, 2π -periodic version of which is naturally defined via a random Fourier series of the form

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [v_n e^{int} + v_n^* e^{-int}] \quad (7)$$

where v_n is a set of i.i.d. complex Gaussian variables with zero mean and the variance $\overline{|v_n|^2} = 1$, with the asterisk standing for the complex conjugation, and the bar for the statistical averaging. It implies the following covariance structure of the random signal

$$\overline{V(t_1)V(t_2)} = -2 \ln |2 \sin \frac{t_1 - t_2}{2}|, \quad t_1 \neq t_2 \in [0, 2\pi) \quad (8)$$

Mathematically such series represents the periodic version of the fractional Brownian motion with the Hurst index $H = 0$. The corresponding definition is formal, as the series in (7) does not converge pointwise, the fact reflected, in particular, in the logarithmic divergence of the covariance in (8)¶. Although it is possible to provide several *bona fide* mathematically correct definitions of the ideal $1/f$ noise as a random generalized function (based, for example, on sampling $2d$ Gaussian free field along the specified curves, e.g. the unit circle for the periodic noise, see [42], or the constructions proposed in [39] or [45]), for all practical purposes the $1/f$ noises should be understood after a proper regularization. In what follows we will use explicitly the regularization proposed by Fyodorov and Bouchaud [30], though we expect the main results must hold, *mutatis mutandis*, for any other regularization.

In the model proposed in [30] one subdivides the interval $t \in (0, 2\pi]$ by finite number M of observation points $t_k = \frac{2\pi}{M}k$ where $k = 1, \dots, M < \infty$, and replaces the function $V(t)$, $t \in [0, 2\pi)$ with a sequence of M random mean-zero Gaussian variables

¶ One can also define other, in general non-periodic versions of similar log-correlated random processes on finite intervals using different basis of orthogonal functions, or even exploit the appropriate random Fourier integral to define the process on the half-line $0 < t < \infty$. The corresponding models arise very naturally in the context of Random Matrix Theory and will be discussed in a separate publication [50]

V_k correlated according to the $M \times M$ covariance matrix $C_{km} = \overline{V_k V_m}$ such that the off-diagonal entries are given by

$$C_{k \neq m} = -2 \ln \left| 2 \sin \frac{\pi}{M} (k - m) \right|. \quad (9)$$

To have a well-defined set of the Gaussian-distributed random variables one has to ensure the positive definiteness of the covariance matrix by choosing the appropriate diagonal entries C_{kk} . A simple calculation [30] shows that as long as we choose

$$C_{kk} = \overline{V_k^2} > 2 \ln M, \quad \forall k = 1, \dots, M. \quad (10)$$

the model is well defined, and we will actually take the minimal possible value: $C_{kk} = 2 \ln M + \epsilon$, $\forall k$ with a small positive $\epsilon \ll 1$. We expect that the statistical properties of the sequence V_k generated in this way reflect correctly the universal features of the $1/f$ noise. An example of the signal generated for $M = 4096$ according to the prescription above via the Fast Fourier Transform (FFT) method as explained in detail in [31] is given in the figure.

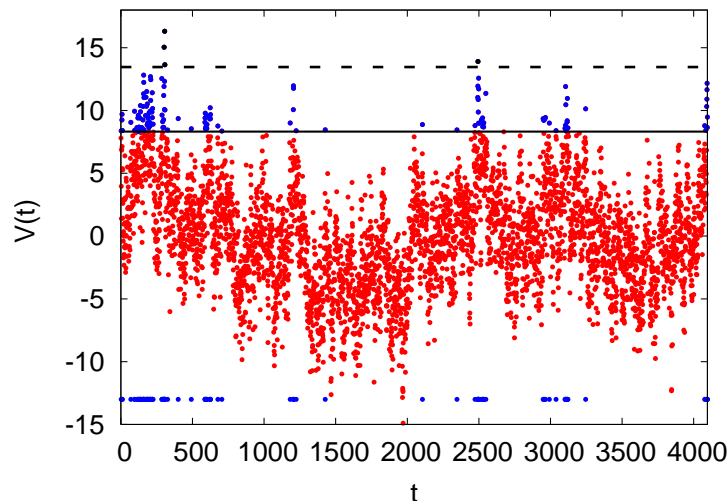


Figure 1. A sample of the regularized periodic $1/f$ noise for $M = 4096$ observation points. The upper broken line marks the typical value of the maximum $V_m = 2 \ln M - \frac{3}{2} \ln \ln M$ and black dots mark a few exceedances of that level. The lower line is at the level $\frac{1}{\sqrt{2}} V_m$ and the blue dots at the bottom mark points i supporting $V_i > \frac{1}{\sqrt{2}} V_m$. The set looks like a fractal.

Using such a model, the authors of [30] defined the associated random energy model via the partition function $Z(\beta) = \sum_{i=1}^M e^{-\beta V_i}$, with the temperature $T = \beta^{-1} \geq 0$ and succeeded in determining the distribution of $Z(\beta)$ in the range $\beta < 1$. To reinterpret those findings in the context of multifractality we introduce the height

variables $h_i = e^{V_i} > 0$ and rename $\beta \rightarrow -q$, converting $Z(\beta)$ of the random energy model to Z_q of the "thermodynamic formalism", eq.(3). Note that due to the statistical equivalence of V_i and $-V_i$ in the model all results may depend only on $|q|$. Then the findings of [30] can be summarized as follows. The probability density of the random variable $Z_{|q|<1}$ consists of two pieces, the *body* and the *far tail*. The body of the distribution has a pronounced maximum at $Z \sim Z_e(q) = M^{1+q^2}/\Gamma(1-q^2) \ll M^2$, and a powerlaw decay when $Z_e \ll Z \ll M^2$. Introducing $z = Z_q/Z_e(q)$ the probability density of such variable is given explicitly by

$$\mathcal{P}_q(z) = \frac{1}{q^2} \left(\frac{1}{z}\right)^{1+\frac{1}{q^2}} e^{-(\frac{1}{z})^{\frac{1}{q^2}}}, \quad z \ll M^{1-q^2}, \quad |q| < 1 \quad (11)$$

For $z \gg M^{1-q^2}$ the above expression is replaced by a lognormal tail[30]. Note that the probability density (11) is characterized by the moments

$$\overline{z^s}_{M \gg 1} \approx \Gamma(1 - sq^2), \quad \text{Re } s < q^{-2} \quad (12)$$

where $\Gamma(z)$ is the Euler Gamma-function. It is worth noting that although the particular form of the density (11) is specific for the chosen model of $1/f$ noise, the power-law forward tail $\mathcal{P}_q(Z) \propto Z^{-1-\frac{1}{q^2}}$ is expected to be universal [31] and so the divergence of moments of the partition function for $\text{Re } s > q^{-2}$. A closely related fact which can be also traced back to the existence of the universal forward tail is that the typical scale $Z_e(q)$ behaves for $q \rightarrow 1$ as $Z_e(q)/M^{1+q^2} \sim (1-q) \rightarrow 0$. This property will have important consequences at the level of the counting function.

To get some understanding of how the above asymptotic results agree with the direct numerical simulations of the model for large, but finite M it is useful to provide the exact finite- M expression for the second moment of the partition function[30]

$$\overline{Z_q^2} = M^{1+4q^2} + M^{2(1+q^2)} S_2^{(M)}(q^2), \quad S_2^{(M)}(q^2) = \frac{1}{M} \sum_{l=1}^{M-1} \left[4 \sin^2 \left(\frac{\pi}{M} l \right) \right]^{-q^2} \quad (13)$$

Second term here is dominant in the large- M limit and gives precisely the result (12) as asymptotically we can replace the sum by the integral convergent for $q^2 < 1/2$ and get

$$\lim_{M \rightarrow \infty} S_2^{(M)}(q^2) = \frac{2}{\pi} \int_0^{\pi/2} [4 \sin^2 \theta]^{-q^2} d\theta = \frac{\Gamma(1-2q^2)}{\Gamma^2(1-q^2)} \quad (14)$$

The main correction to this asymptotic result is given by the first term in (13), whose relative contribution is small as M^{1-2q^2} for $q^2 < 1/2$. In figure 2 we show the exact evaluation for the relative variance $\delta_z(q, M) = \frac{\overline{z^2}}{(\overline{z})^2} - 1$ with $z = Z_q/Z_e(q)$ given explicitly by

$$\delta_z(q, M) = S_2^{(M)}(q^2) + M^{2q^2-1} - 1 \quad (15)$$

We see that $\delta_z(q, M)$ clearly approaches the asymptotic value $\delta_z(q, \infty) = \frac{\Gamma(1-2q^2)}{[\Gamma(1-q^2)]^2} - 1$ for $M \sim 2^{11}$.

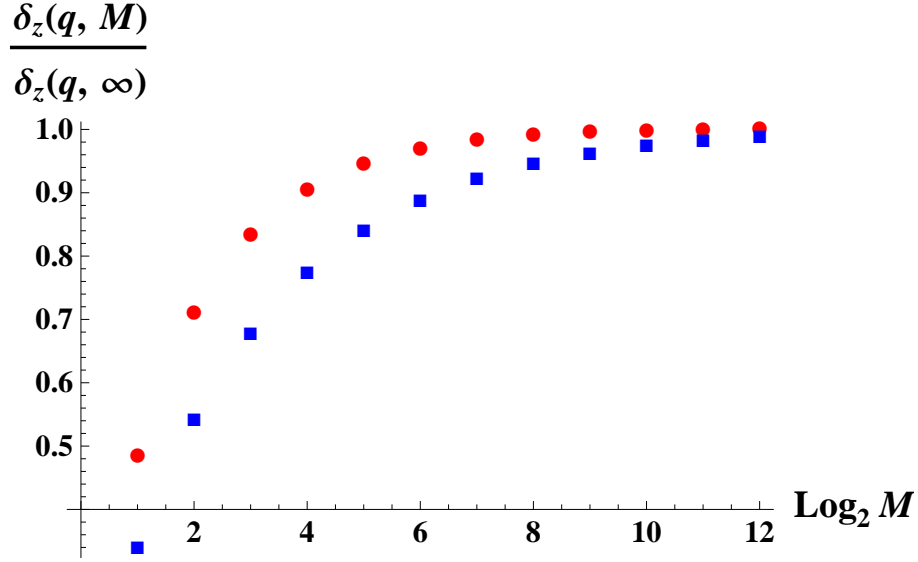


Figure 2. Convergence of the relative variance to the asymptotic value. Red dots correspond to $q = 1/3$ and blue dots correspond to $q = 1/2$. The asymptotic behavior is reached for $M \sim 2^{11}$. When q increases the convergence slows down.

3. Thermodynamic formalism for the counting function of the $1/f$ noise sequence and the threshold of extreme values

The statistics of Z_q for the model under consideration suggest that it reflects the corresponding strong sample-to-sample fluctuations in the counting function of pattern of heights. Our goal is to quantify statistics of those fluctuations by considering the total number $N_>(x)$, which in the present context will be denoted as $\mathcal{N}_M(x)$, of the x -high points in the (regularized) $1/f$ sequence V_1, \dots, V_M such that $V_i > x \ln M$ which is equivalent to $h_i = e^{V_i} > M^x$.

We then relate the number $\mathcal{N}_M(x)$ to the partition function Z_q by the thermodynamic formalism:

$$\mathcal{N}_M(x) = \ln M \int_x^\infty \tilde{\rho}_M(y) dy, \quad Z_q = \ln M \int_{-\infty}^\infty M^{qy} \tilde{\rho}_M(y) dy \quad (16)$$

where now the density $\tilde{\rho}_M(y) = \frac{\rho_M(y)}{\ln M} = \sum_{k=1}^M \delta(V_k - y \ln M)$ is anticipated to be given in the large- M limit by the multifractal ansatz of the particular "improved" form

$$\tilde{\rho}_M(y) \approx c_M(y) \frac{M^{1-y^2/4}}{\sqrt{\ln M}}, \quad c_M(y) = \frac{n_M(y)}{2\sqrt{\pi}\Gamma(1-y^2/4)}, \quad |y| < 2 \quad (17)$$

with a random coefficient $n_M(y)$ of order of unity which strongly fluctuates from one realization of the sequence V_i to the other in such a way that its probability density is given by (11) with $q = y/2$. Indeed, substituting such form of the density

to Z_q in (16) and performing the integrals in the limit $\ln M \gg 1$ by the Laplace method we arrive at the asymptotic behaviour $Z_q \approx n_M(2q) Z_e(q)$, $q < 1$, where $Z_e(q) = \frac{M^{1+q^2}}{\Gamma(1-q^2)}$ precisely as we have found from the exact solution (11). On the other hand, substituting the same ansatz to the counting function in (16) yields by the same method $\mathcal{N}_{M \gg 1}(x) \approx n_M(x) \mathcal{N}_t(x)$ where the typical value $\mathcal{N}_t(x)$ is given by

$$\mathcal{N}_t(x) = \frac{M^{1-x^2/4}}{x\sqrt{\pi \ln M}} \frac{1}{\Gamma(1-x^2/4)}, \quad 0 < x < 2. \quad (18)$$

Thus, our main conclusion is that the random variables $n = \mathcal{N}_M(x)/\mathcal{N}_t(x)$ and $z = Z_{q/2}/Z_e(q/2)$ must be distributed in the large- M limit according to the same probability law, which after invoking (11) yields the asymptotic probability density for the scaled counting function in the form

$$\mathcal{P}_x(n) = \frac{4}{x^2} \frac{1}{n^{1+\frac{4}{x^2}}} e^{-(\frac{1}{n})^{\frac{4}{x^2}}}, \quad 0 < x < 2. \quad (19)$$

The shape of the distribution for a few values of x is presented in Fig. 3.

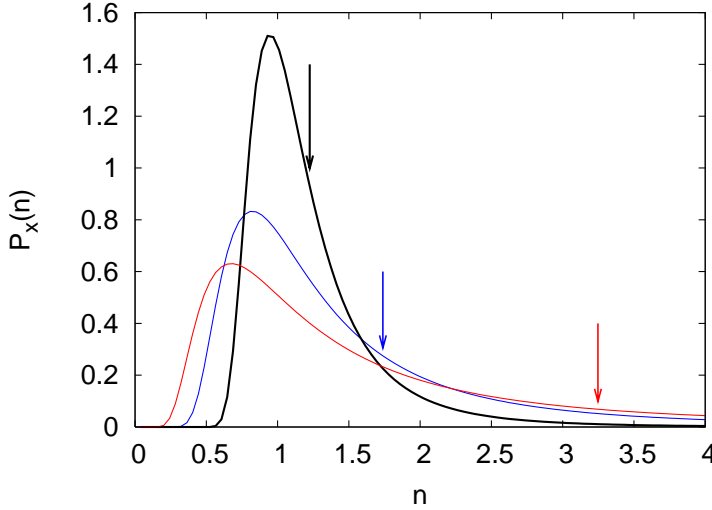


Figure 3. From top to bottom the probability density Eq.(19) for $x = 1$, $x = 1.4$, $x = 1.7$. Arrows indicates the position of the mean, the typical value corresponds always to 1 in the chosen scaling.

The following qualification is needed here. For large but finite M such form of the density stops to hold true for extremely large $n \rightarrow \infty$ as in any realization obviously $\mathcal{N}_M(x) < M$ at the very least. Therefore there must exist an upper cut-off value $\mathcal{N}_c(x)$ such that for $n > n_c = \mathcal{N}_c(x)/\mathcal{N}_t(x)$ the scaling form of the probability density (19) loses its validity. Similarly, another restriction on the validity of (19) should exist in the region of extremely small $n \rightarrow 0$ due to implicit condition $\mathcal{N}_M(x) \gg 1$. Precise value of the cutoffs can not be extracted in the framework of the thermodynamic formalism,

but in the end of the paper we will get an estimate for n_c in the range $\sqrt{2} < x < 2$ from independent calculations, see (32).

The important scale $\mathcal{N}_t(x)$ defined explicitly in (18) describes *typical* values of the counting function $\mathcal{N}_M(x)$ for a given observation level x . In particular, it can be used to define one of the objects of central interest in the present paper, the *threshold of extreme values*. The latter stands for such a level above which typically we can find for $\ln M \gg 1$ only a few, i.e. of the order of one points of our random sequence. The scaling behaviour $\mathcal{N}_t(x) \sim M^{f(x)}$, $f(x) = 1 - x^2/4$ is the hallmark of the multifractality. A very similar parabolic singularity spectrum characterizes the high value pattern of the two-dimensional Gaussian free field as revealed in [18] and proved in a mathematically rigorous way in [44]. The above result for $f(x)$ is the simple one-dimensional analogue of that fact, see e.g. [45] where it is rigorously shown that $\lim_{M \rightarrow \infty} \frac{\ln \mathcal{N}_M(x)}{\ln M} = 1 - x^2/4$ in our notations. Note that as the singularity spectrum $f(x)$ vanishes at $x = 2$ the typical position of the absolute maximum of the random sequence of V_i 's is given by $V_{max} = 2 \ln M$ at the leading order. The corresponding subleading term was conjectured in the work by Carpentier and Le Doussal [29] to be $V_m = 2 \ln M - c \ln \ln M + O(1)$, with $c = 3/2$. That conclusion was based upon an analysis of the travelling wave-type equation [51] appearing in the course of one-loop renormalization group calculation, and the value $c = 3/2$ was conjectured to be universally shared by all systems with logarithmic correlation. Such a result is markedly different from $c = 1/2$ typical for short-ranged correlated random signals, so the value of c may be used as a sensitive indicator of the universality class. Indeed, in a recent numerical studies of the behaviour of the logarithm of the modulus of the Riemann zeta-function along the critical line [47] the value $3/2$ was used to confirm the consistency of describing that function as a representative of logarithmically correlated processes. Despite its importance, no transparent qualitative argument explaining $c = 3/2$ vs. $c = 1/2$ values was ever provided, to the best of our knowledge, though for the case of the 2D Gaussian free field the value $3/2$ was very recently rigorously proved by Bramson and Zeitouni [52] by exploiting elaborate probabilistic arguments. Below we suggest a very general and transparent argument showing that the change from $c = 1/2$ to $c = 3/2$ is a direct consequence of the strong fluctuations in the counting function reflected in the power-law decay of the probability density (19). That observation not only allows one to explain the origin of $c = 3/2$ for the Gaussian case, but has also a predictive power in a more general situation as will be demonstrated in the section 5.

To begin with presenting the essence of our argument we first observe that (19) implies that $\bar{n} = \Gamma\left(1 - \frac{x^2}{4}\right)$, $0 < x < 2$ so that the mean value of the counting function is given asymptotically by

$$\overline{\mathcal{N}_M(x)} \approx \frac{M^{1-x^2/4}}{x\sqrt{\pi \ln M}}, \quad (20)$$

We shall see in the next section that the above expression is asymptotically exact for any real $x > 0$ without restriction to $0 < x < 2$. Notice however that the mean value (20) and the characteristic scale $N_t(x)$ differ from each other by the factor $\frac{1}{\Gamma(1-x^2/4)}$ tending to zero as $x \rightarrow 2$. Such a difference, origin of which can be again traced to the specific power-law tail of the probability density (19), see Fig. 3 and is one of the hallmarks of the random signals with logarithmic correlations. Indeed, consider for comparison the case of uncorrelated i.i.d. Gaussian sequence sharing the same variance $\overline{V_i^2} = 2 \ln M$ with the logarithmically correlated noise (by historical reasons it is natural to refer to such model as the Random Energy Model, or REM [53]). A straightforward calculation shows that we still would have precisely the same mean value of the counting function as in the logarithmic model, but unlike the latter it will be simultaneously the typical value of that random variable as no powerlaw tail is present in that case (see Fig. 4 and discussion in the next section).

Such a difference between the two cases has important implications for the location of the threshold $x = x_m$ which corresponds to the region of extreme value statistics of multifractal heights. Indeed, by approximating the singularity spectrum close to its right zero $x_+ = 2$ as $f(x) \approx (2 - x)$, and similarly writing $\Gamma\left(1 - \frac{x^2}{4}\right) \propto (2 - x)^{-1}$ we observe that the condition $\mathcal{N}_t(x_m) \sim 1$ is equivalent to the equation:

$$(2 - x_m) \ln M - \frac{1}{2} \ln \ln M + \ln(2 - x_m) = 0 \quad (21)$$

solving which for $\ln M \gg 1$ to the first non-trivial order gives precisely $x_m = 2 - c \frac{\ln \ln M}{\ln M}$ with $c = 3/2$. Had we replaced in the above condition the typical value $\mathcal{N}_t(x_m)$ with the mean $\overline{\mathcal{N}_M(x_m)}$ we would arrive to the same expression for x_m but with the value $c = 1/2$ replacing $c = 3/2$. This perfectly agrees with such x_m being the extreme value threshold for short-ranged correlated random sequences. The suggested explanation of the transmutation of the coefficient c based on the "typical versus mean" argument seems to us very transparent and supports the conjectured universality of the result. Indeed, the thermodynamic formalism combined with the results of [31] suggests that the power-law forward tail $\mathcal{P}_x(\mathcal{N}) \propto \frac{1}{N} \left(\frac{N_\epsilon}{N}\right)^{\frac{4}{x^2}}$ should be universal for one-dimensional Gaussian processes with logarithmic correlations. It also should show up, *mutatis mutandis*, in higher-dimensional versions of the model like e.g. the Gaussian Free Field on the lattice. Such a tail will ensure the difference between the typical and the mean of the counting function by a factor which vanishes linearly on approaching the extreme value threshold value $x = x_m$. This factor then will lead to the value $c = 3/2$ by the mechanism illustrated above for our particular explicit example. The case of non-Gaussian signals with logarithmic correlations is relevant for general disorder-generated multifractals and is discussed in Section 5 of the paper.

Finally it is worth mentioning that the fact that for $x > 2$ the mean value of the counting function (20) is exponentially small reflects the need to generate exponentially

large number of samples to have at least a single event with $V_i > 2x \ln M$ for $\ln M \gg 1$ as such values of V_i will not show up in a typical realization (cf. earlier discussion about "annealed" vs. "quenched" singularity spectra).

4. Exact results versus asymptotics

The above results for the counting function obtained in the framework of the thermodynamic formalism are expected to be valid as long as $\ln M \gg 1$. To get a feeling of how big in practice should be M to ensure the validity of our asymptotic formulae, it is natural again to try to perform the direct numerical simulations of the regularized version of the ideal $1/f$ noise. We start with checking directly the distribution of the scaled counting function, Eq.(19), for a particular values $x = 1$. The results are presented in Fig.4. They show that although the main qualitative features of the distribution (in particular, the well-developed powerlaw tail) are clearly in agreement with the theoretical predictions, the curve is still rather far from its predicted asymptotic shape for $M = 2^{18}$, and the convergence is too slow to claim a quantitative agreement.

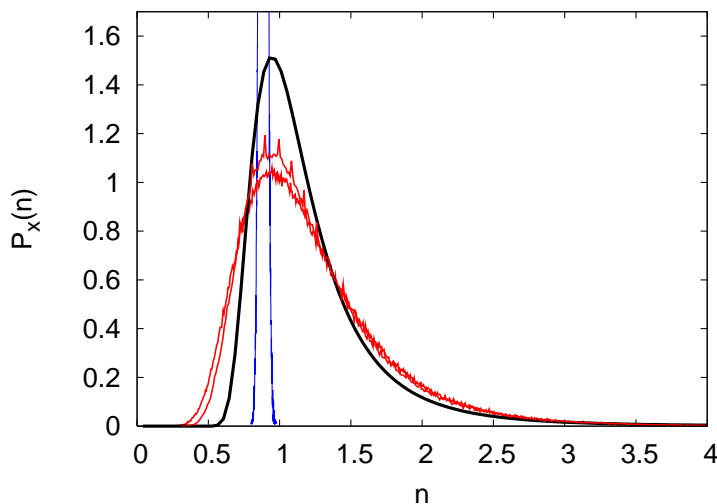


Figure 4. Probability density for the scaled counting function, $\mathcal{P}_x(n)$ with $x = 1$. Black solid line corresponds to the analytical prediction Eq.19. Red lines correspond to numerical simulation of the regularized $1/f$ Gaussian signal generated via the Fast Fourier Transform (FFT) method (lower red curve $M = 2^{14}$, higher $M = 2^{18}$, data collected from 10^5 samples.). The blue line corresponds to the REM model: $M = 2^{18}$ i.i.d. zero mean Gaussian variables with variance $2 \ln M$; the density for $n = \mathcal{N}_M(x)/\overline{\mathcal{N}_M(x)}$ is clearly converging to the delta peak and does not show any power-law tails, see discussion in the text.

To get a better understanding of the mechanism of such disagreement at a

quantitative level, and to check the results obtained in the framework of the thermodynamic formalism we choose to consider in much greater detail the first two moments of the counting function. The asymptotic formula (19) yields for the mean of the counting function the expression (20) and for the variance

$$\frac{[\overline{\mathcal{N}_M(x)}]^2 - [\overline{\mathcal{N}_M(x)}]^2}{[\overline{\mathcal{N}_M(x)}]^2} \approx \frac{\Gamma(1 - x^2/2)}{\Gamma^2(1 - x^2/4)} - 1, \quad 0 < x < \sqrt{2} \quad (22)$$

At the same time it is possible to derive a closed form exact finite M expression for the first two moments of the counting function $\mathcal{N}_M(x) = \sum_{i=1}^M \theta(V_i - x \ln M)$ without any recourse to the thermodynamic formalism. Here we have used the Heaviside step function $\theta(u) = 1$ for $u > 0$ and $\theta(u) = 0$ otherwise. The mean value can be immediately computed as

$$\overline{\mathcal{N}_M(x)} = \frac{M}{2\sqrt{\pi \ln M}} \int_{x \ln M}^{\infty} \exp\left(-\frac{v^2}{4 \ln M}\right) dv = \frac{M}{2} \operatorname{Erfc}\left(\frac{x}{2} \sqrt{\ln M}\right) \quad (23)$$

and is independent of the correlations. The problem of deriving a closed-form expression for the variance which is amenable to accurate numerical evaluation for very big $\ln M \gg 1$ is less trivial and may have an independent interest. Before presenting the results we find it most convenient to define the following object

$$\Delta_M(x) = \overline{N_M(x) (N_M(x) - 1)} - \frac{M - 1}{M} [\overline{N_M(x)}]^2 \quad (24)$$

in terms of which the relative variance is expressed as

$$\frac{[\overline{\mathcal{N}_M(x)}]^2 - [\overline{\mathcal{N}_M(x)}]^2}{[\overline{\mathcal{N}_M(x)}]^2} = \frac{1}{M} + \frac{1}{\overline{\mathcal{N}_M(x)}} + \delta_n(x; M), \quad \delta_n(x; M) = \frac{\Delta_M(x)}{(\overline{\mathcal{N}_M(x)})^2} \quad (25)$$

$\Delta_M(x)$ is a convenient measure of correlation-induced fluctuations. Using the definition of the counting function we explicitly get:

$$\Delta_M(x) = \sum_{i \neq j} \left[\overline{\theta(V_i - x \ln M) \theta(V_j - x \ln M)} - \overline{\theta(V_i - x \ln M)} \cdot \overline{\theta(V_j - x \ln M)} \right] \quad (26)$$

from which is evident that $\Delta_M(x)$ vanishes for any i.i.d. sequence. In the latter case the relative variance tends to zero in the large- M limit assuming $\overline{\mathcal{N}_M(x)} \rightarrow \infty$ for $M \rightarrow \infty$. This simply means that in the i.i.d. case the variable $\mathcal{N}_M(x)/\overline{\mathcal{N}_M(x)}$ is self-averaging, i.e its limiting density approaches the Dirac delta-function, see Fig. 4. On the other hand, $\Delta_M(x)$ is formally different from zero for correlated variables. Nevertheless, using the general formalism exposed in the Appendix A one can satisfy oneself that for stationary Gaussian sequences with correlations decaying fast enough to zero at big separations (e.g. as a power of the distance) the quantity $\delta_n(x; M)$ still tends to zero as $M \rightarrow \infty$. In contrast, we will see below that in the logarithmically correlated case

$\delta_n(x; M)$ tends to a finite positive number for $M \rightarrow \infty$, and thus coincides with the leading behaviour of the variance of the counting function.

In the Appendix A we have derived the exact expression for $\Delta_M(x)$ for a general correlated Gaussian sequence. For the periodic $1/f$ noise sequence using (9) the result reads

$$\Delta_M(x) = \frac{1}{\pi} M^{1-x^2/4} \sum_{n=1}^{M-1} \int_{h_n}^1 \frac{d\tau}{1+\tau^2} e^{-\ln M x^2 \tau^2/4}, \quad h_n = \sqrt{\frac{\ln M + \ln |2 \sin(\frac{\pi n}{M})|}{\ln M - \ln |2 \sin(\frac{\pi n}{M})|}} \quad (27)$$

In the figure we test its validity by comparing the results obtained for moderate value of M by direct numerical simulations of the log-correlated sequences with the predictions of (27).

The expressions above are suitable for developing a well-controlled approximation to the exact expression (27) in the large- M limit assuming $\ln M \gg 1$. First of all, it is clear that in the large- M limit we may replace the discrete sum in (27) by the integral treating $\frac{\pi n}{M}$ as a continuous variable θ . Using the symmetry $\theta \rightarrow \pi - \theta$ we then arrive to an approximation to the ratio $\delta_n(x; M) = \frac{\Delta_M(x)}{(\mathcal{N}_M(x))^2}$ given by

$$\delta_n(x; M) \approx \frac{8M^{-x^2/4}}{\pi^2 \text{Erfc}^2\left(\frac{1}{2}x\sqrt{\ln M}\right)} \int_{\frac{\pi}{M}}^{\frac{\pi}{2}} d\theta \int_{h_\theta}^1 \frac{d\tau}{1+\tau^2} e^{-\ln M x^2 \tau^2/4}, \quad h_\theta = \sqrt{\frac{\ln M + \ln |2 \sin \theta|}{\ln M - \ln |2 \sin \theta|}} \quad (28)$$

At the next step we assume that the θ -integral is dominated by the finite values $0 < \theta < \pi/2$. This allows to replace the lower limit of integrations by 0 and also to expand $h_\theta \approx 1 + \ln |2 \sin \theta| / \ln M$ for $\ln M \rightarrow \infty$. Changing then the integration variable $\tau = (1 + u / \ln M)^{1/2}$ and keeping only the leading order terms the u -integral is easily calculated and the emerging θ -integral takes the form $\int_0^\pi (4 \sin^2 \theta)^{-x^2/4} d\theta$. The latter is convergent as long as $x < \sqrt{2}$ where it can be reduced to the Euler beta-function, see (14). As a result, we arrive to the following expression:

$$\lim_{M \rightarrow \infty} \delta_n(x; M) = \frac{\Gamma(1 - x^2/2)}{\Gamma^2(1 - x^2/4)} - 1, \quad 0 < x < \sqrt{2} \quad (29)$$

which is fully equivalent to the variance result (22) we have anticipated on the basis of the thermodynamic formalism. For $x > \sqrt{2}$ the θ -integral diverges at the lower limit $\theta \rightarrow 0$ rendering our large- M procedure invalid. This case will be separately treated in the end of the section.

Now we are in a position to check numerically the range of applicability of the approximations derived. First we attempt to compare the results of exact numerical evaluation of the discrete sum in (27) to the integral (28). Actually, the direct evaluation of the sum in Mathematica is affordable up to $M = 50000$. To go up to the higher

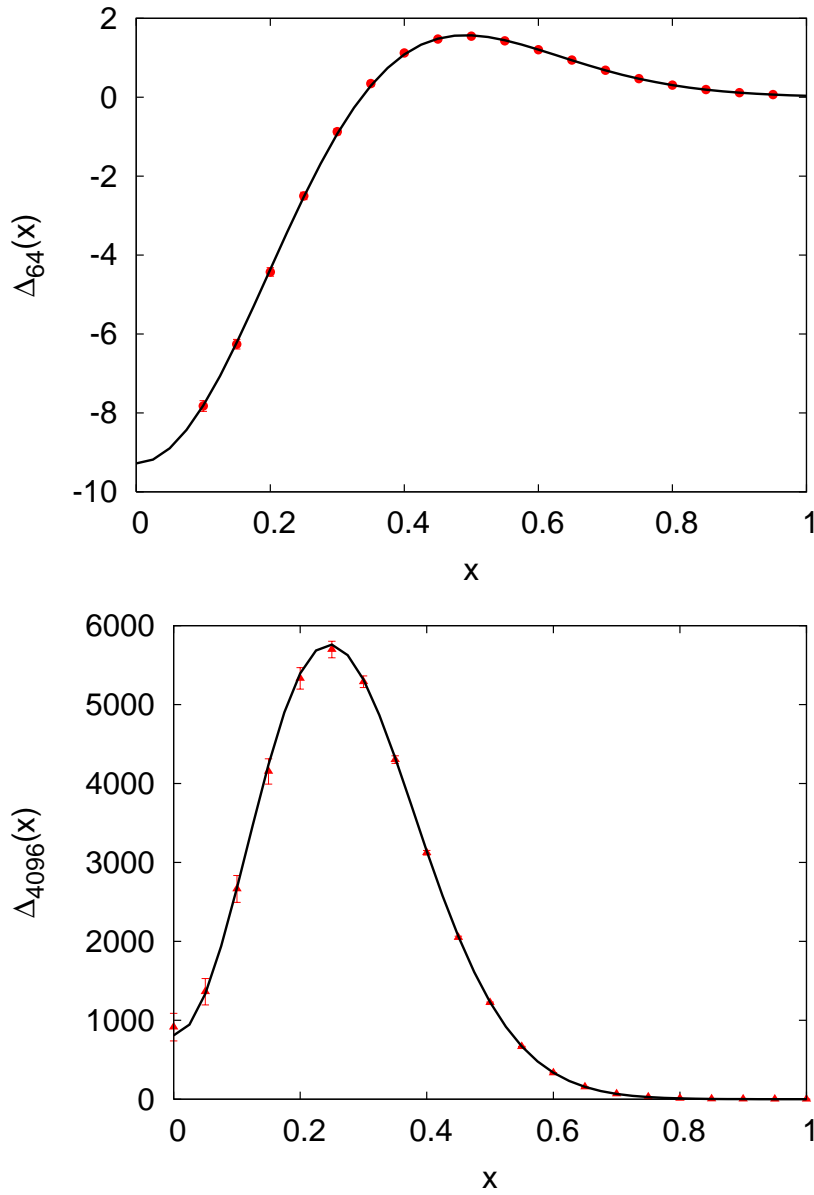


Figure 5. Numerical simulations (red bars) compared to the theoretical prediction (27) (black line) for $M = 64$ (top) and $M = 4096$ (bottom). $1/f$ signal is generated via the Fast Fourier Transform (FFT) method.

values of M we use the identity

$$\sum_{n=1}^{M-1} \int_{h_n}^1 f(\tau) d\tau = \sum_{n=1}^{\frac{M}{2}-1} 2R(n) \int_{h_n}^{h_{n+1}} f(\tau) d\tau \quad (30)$$

where $R(n) = n$ for $n < M/6$ and $R(n) = n - M/3 + \frac{1}{2}$ for $n \geq M/6$. Since for large $\ln M$ the difference $|h_n - h_{n+1}|$ is very small, the integral in the above expression can

be very accurately approximated by the trapezoidal rule

$$2 \int_{h_n}^{h_{n+1}} f(\tau) d\tau \approx (h_{n+1} - h_n) (f(h_{n+1}) + f(h_n)) .$$

This trick allows us to evaluate the sum in (27) numerically up to $M \sim 10^9$.

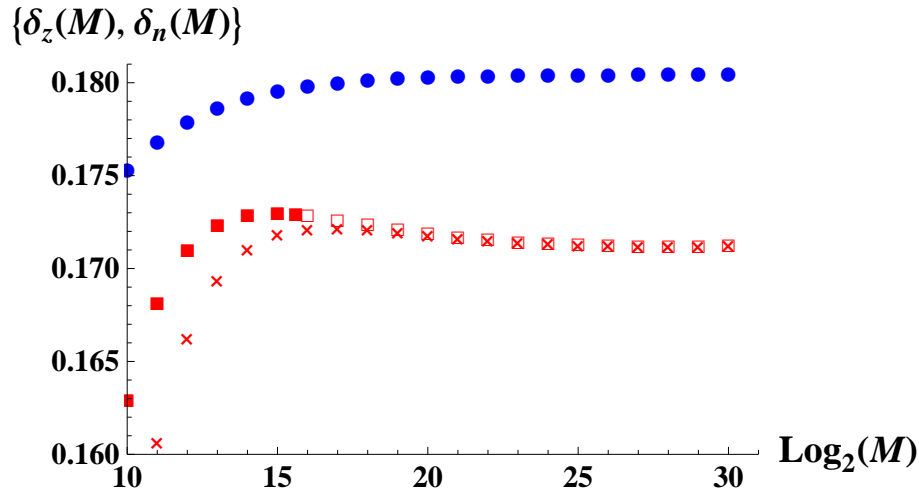


Figure 6. Comparison of the convergence to the asymptotic values. The partition function variance (blue circles , $\delta_z(q = 0.5, M)$) reaches the asymptotic value $\delta_z(q = 0.5, \infty) = 0.180341$ for $M \sim 2^{16}$. The relative variance $\delta_n(x = 1, M)$ for the counting function is presented as evaluated by three different methods: Filled Squares stand for the result of precise integration of the discrete sum (27) up to $M = 50000$. Empty Squares stand for a numerical integration involving the identity (30) and the trapezoidal rule. Diamonds describe the continuum approximation Eq.(28). The asymptotic value predicted by (29) for $M \rightarrow \infty$ is again $\delta_n(x = 1; \infty) = 0.180341$ is nowhere close to the finite- M results and the agreement even seems to worsen with growing M . See the next figure.

The results are presented figure 5 and figure 6 for $x = 1$. In figure 5 we compare the fast convergence of $\delta_z(q = 0.5, M)$ against the very slow convergence of $\delta_n(x = 1; M)$: for $M \sim 2^{15}$ $\delta_z(q = 0.5, M)$ is already very close to the asymptotic value whereas $\delta_n(x = 1, M)$ shows a curious non-monotonicity and even for $M \sim 2^{30}$ we are still very far from the asymptotic value predicted by (29) which is $\delta_\infty^{(n)}(x = 1) = 0.180341$. We observe that for M larger than 2^{17} the continuum approximation of Eq.(28) matches perfectly the numerical integration involving the identity (30) and the trapezoidal rule. The latter matches perfectly the exact discrete sum (27) even for moderate M . As we now are confident in the accuracy of the continuum approximation formula (28) given by a double integral with $\ln M$ entering as a simple parameter we can use it to check what values of $\ln M$ actually ensure the validity of the asymptotic (29). The results are

presented in figure 6 and show that one needs astronomically big values of M even to achieve a rather modest agreement with the asymptotic value.

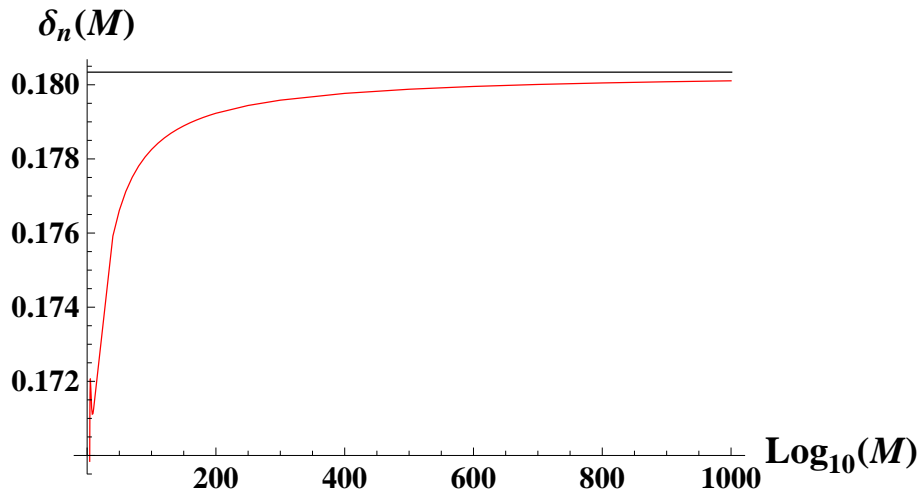


Figure 7. $\delta_n(x = 1, M)$ given by the continuum approximation formula (28). The non-monotonic region seen in the previous figure is reflected in a tiny dip confined to a region close to the origin. The asymptotic value predicted by (29) is $\delta_n(x = 1; \infty) = 0.180341$.

These facts explain our failure to confirm the infinite- M asymptotic by direct simulation of the variance of the counting function predicted for the periodic $1/f$ noise in the framework of the thermodynamic formalism. In any realistic signal analysis of such variance one must therefore rely upon the exact formula (27) or its analogues instead of the asymptotic value (29). The conclusion should be of significant practical importance, in particular in view of the growing interest in numerical investigations of statistical properties of high values of the modulus of the Riemann zeta-function and of the characteristic polynomials of large random matrices.

Having verified by the independent approach the asymptotic of the second moment of the counting function for $x < \sqrt{2}$ we can now apply a similar method beyond that range. The formal divergence of the right-hand side in (29) for $x \rightarrow \sqrt{2}$ simply means that the second moment of the counting function is not proportional to the squared typical scale \mathcal{N}_e , but is parametrically larger and is actually determined by the detail of the probability density (19) close to its upper cut-off n_c which is not yet known to us. An accurate analysis of the second moment in the range $\sqrt{2} < x < 2\sqrt{2}$ performed in the Appendix B shows that the leading asymptotic behaviour of $\delta_n(x; M)$ is given by

$$\delta_n(x; M) \approx x^2 \sqrt{\frac{\sqrt{2} \pi \ln M}{[2 - x/\sqrt{2}]}} M^{\left(\frac{x}{\sqrt{2}} - 1\right)^2}, \quad \sqrt{2} < x < 2\sqrt{2} \quad (31)$$

As is easy to see $\delta_n(x; M) \gg \left(\overline{N_M(x)}\right)^{-1}$ in the above domain, and is therefore the dominant term in the relative variance (25). The above result can be used to estimate the upper cutoff parameter n_c . Indeed, as $\delta_n(x; M) \gg 1$ the formula (25) in the range $\sqrt{2} < x < 2$ can be rewritten $\overline{n^2} \approx \Gamma\left(1 - \frac{x^2}{4}\right) \delta_n(x; M)$, with $\delta_n(x; M)$ given by (31). On the other hand, in that range of x the probability density (19) implies $\overline{n^2} \sim n_c^{2 - \frac{4}{x^2}}$. Comparing the two results and using (31) yields the scaling of the upper cutoff:

$$n_c \sim M^{\frac{x^2}{4} \frac{x - \sqrt{2}}{x + \sqrt{2}}}, \quad \sqrt{2} < x < 2 \quad (32)$$

5. The position of threshold of extreme values in generic disorder-generated multifractal patterns

Results obtained so far in the paper suggest a natural question about statistics of high values and positions of extremes of more general power-law correlated multifractal random field with a general non-parabolic singularity spectrum. Most obvious examples include the variety of the Anderson transitions[14], but in fact many more random critical systems should be of that sort[10]. A straightforward calculation outlined in [22] shows that behind each pattern of such type lurks a certain logarithmically correlated field, though in general of a non-Gaussian nature. Below we sketch that simple argument for the sake of completeness. Consider a d -dimensional sample of linear size L , and assume following [10] that the multifractal patterns of intensities $p(\mathbf{r})$ is *self-similar*

$$\overline{p^q(\mathbf{r}_1)p^s(\mathbf{r}_2)} \propto L^{-y_{q,s}} |\mathbf{r}_1 - \mathbf{r}_2|^{-z_{q,s}}, \quad q, s \geq 0, \quad a \ll |\mathbf{r}_1 - \mathbf{r}_2| \ll L \quad (33)$$

and *spatially homogeneous*

$$\overline{p^q(\mathbf{r}_1)} = \frac{1}{L^d} I_q, \quad \text{where} \quad I_q = \int_{|\mathbf{r}| < L} p^q(\mathbf{r}) d\mathbf{r} \propto L^{-\tau_q} \quad (34)$$

The consistency of the above two conditions for $|\mathbf{r}_1 - \mathbf{r}_2| \sim a$ and $|\mathbf{r}_1 - \mathbf{r}_2| \sim L$ implies:

$$y_{q,s} = d + \tau_{q+s}, \quad z_{q,s} = d + \tau_q + \tau_s - \tau_{q+s} \quad (35)$$

If we now introduce the field $V(\mathbf{r}) = \ln p(\mathbf{r}) - \overline{\ln p(\mathbf{r})}$ and combine the identity $\frac{d}{ds} p^s|_{s=0} = \ln p$ with the fact that $\tau_0 = -d$ we straightforwardly arrive at the relation

$$\overline{V(\mathbf{r}_1)V(\mathbf{r}_2)} = -g^2 \ln \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{L}, \quad g^2 = \frac{\partial^2}{\partial s \partial q} \tau_{q+s}|_{s=q=0} \quad (36)$$

Thus we conclude that the logarithm of any multifractal intensity is a log-correlated random field. The above argument does not say anything about higher cumulants of the field $V(\mathbf{r})$, but it is easily checked that had those fields be always Gaussian the resulting singularity spectrum $f(\alpha)$ obtained from τ_q via the Legendre transform

would be invariably parabolic. Therefore, any non-parabolicity of the singularity spectra necessarily implies non-Gaussian nature of the underlying log-correlated fields. Nevertheless, combining our previous insights with results for disorder-generated multifractal patterns in [14] suggests the way in which our results on Gaussian $1/f$ noise can be generalized to statistics of high values and positions of extremes of more general non-Gaussian logarithmically correlated random processes and fields.

As was already mentioned in the introduction, in the case of the Anderson transition the probability density of the inverse participation ratios I_q depends only on the scaling ratio $z = I_q/I_q^{(t)}$, with $I_q^{(t)}$ standing for the typical value. Moreover, that ratio is expected to be power-law distributed: $\mathcal{P}_q(z) \sim z^{-1-\omega_q}$ [14, 9]. We may try to combine that fact with the theory developed in the present paper to conjecture the typical position of the extreme values (maxima or minima) in a pattern of multifractal probability weights $p_i \sim M^{-\alpha_i}$ $i = 1, \dots, M \sim L^d$. A brief account of such a procedure is as follows. Suppose the mean participation ratios are given by $\overline{I}_q = B(q) M^{-\tau_q}$, with a coefficient $B(q)$ of order of unity, and concentrate on those q for which typical and annealed exponents coincide. From it we recover in the usual way the singularity spectrum $f(\alpha)$ by the corresponding Legendre transform: $f(\alpha) \geq 0$ for $\alpha \in [\alpha_-, \alpha_+]$ and further assume $\alpha_- > 0$ to avoid complications related to the so-called "multifractality freezing" [9, 13, 20] which would require a special care. Define $\alpha(q)$ as the solution of equation $q = f'(\alpha)$ and denote $\overline{z}_q = \int_0^\infty \mathcal{P}_q(z) z dz$. Further given any function ϕ_q of the variable q define a "Lagrange conjugate" function $\phi_*(\alpha)$, by the relation $\phi_*(\alpha(q)) = \phi_q$. Then we suggest that the density of exponent $\rho_M(\alpha) = \sum_{i=1}^M \delta(\frac{\ln p_i}{\ln M} - \alpha)$ should be given asymptotically by the following "improved" multifractal ansatz:

$$\rho_M(\alpha) = \frac{n_*(\alpha)}{\overline{z}_*(\alpha)} B_*(\alpha) \sqrt{\frac{\ln M |f''(\alpha)|}{2\pi}} M^{f(\alpha)} \quad (37)$$

where $n = n_*(\alpha)$ is a random coefficient of the order of unity distributed for a given α according to the probability density $\mathcal{P}_\alpha^*(n)$ defined in terms of $\mathcal{P}_q(z)$ via the rule $\mathcal{P}_{\alpha(q)}^*(n) = \mathcal{P}_q(n)$. Indeed, substituting such an ansatz into the definition $I_q = \int_0^\infty M^{-q\alpha} \rho_M(\alpha) d\alpha$ and performing the integral by the Laplace method for $\ln M \gg 1$ gives $I_q \approx n_*(\alpha(q)) [\overline{I}_q/\overline{z}_q]$, where the random variable $z = n_*(\alpha(q))$ is distributed according to the probability density $\mathcal{P}_q(z)$. This is precisely what is required, provided we identify $I_q^{(t)} = \overline{I}_q/\overline{z}_q$. Now we can substitute the same ansatz to the definition of the counting function $N_<(\alpha) = \int_{-\infty}^\alpha \rho(\alpha) d\alpha$ choosing α to the left of the maximum of $f(\alpha)$. This gives asymptotically $N_<(\alpha) = n_*(\alpha) N_t(\alpha)$ where the scale $N_t(\alpha)$ is now given by

$$N_t(\alpha) = \frac{B_*(\alpha)}{\overline{z}_*(\alpha) f'(\alpha)} \sqrt{\frac{|f''(\alpha)|}{2\pi \ln M}} M^{f(\alpha)}, \quad \alpha_- < \alpha < \alpha_0 \quad (38)$$

and defines the typical value of the counting function. A few typical "extreme" values among p_i 's in the multifractal pattern will be of the order of $p_m = M^{-\alpha_m}$, where α_m is determined from the condition that the counting function becomes of the order of unity: $N_i(\alpha_m) \sim 1$. Clearly, at the leading order $\alpha_m = \alpha_-$ given by the left root of $f(\alpha) = 0$, and the goal is to extract the subleading term. For doing this a crucial observation taken from [14] is that for $q \rightarrow q_c = f'(\alpha_-)$ the tail exponent ω_q of the IPR probability density $\mathcal{P}_q(z) \sim z^{-1-\omega_q}$ should tend to $\omega_{q_c} = 1$. As the derivative $\omega'_{q=q_c}$ is generically neither zero nor infinity the average \bar{z}_q diverges as $\bar{z}_q \sim (q_c - q)^{-1}$. In turn, as $\bar{z}_*(\alpha)$ is the Lagrange conjugate of \bar{z}_q the divergence of the latter implies similar behaviour $\bar{z}_*(\alpha) \sim (\alpha - \alpha_-)^{-1}$ in the vicinity of α_- . At the same time generically $f'(\alpha), f''(\alpha)$ neither vanish nor diverge at $\alpha = \alpha_-$, and we do not see any reasons to expect that $B_*(\alpha)$ vanishes or diverges at this point either. Approximating $f(\alpha_m) \approx f'(\alpha_-)(\alpha_m - \alpha_-)$ we arrive at the following equation for the extreme value threshold α_m :

$$f'(\alpha_-)(\alpha_m - \alpha_-) \ln M - \frac{1}{2} \ln \ln M + \ln(\alpha_m - \alpha_-) = 0 \quad (39)$$

solving which for $\ln M \gg 1$ to the first non-trivial order gives

$$\alpha_m \approx \alpha_- + \frac{3}{2} \frac{1}{f'(\alpha_-)} \frac{\ln \ln M}{\ln M} \Rightarrow -\ln p_m^{(typ)} \approx \alpha_- \ln M + \frac{3}{2} \frac{1}{f'(\alpha_-)} \ln \ln M \quad (40)$$

which is our main prediction for the typical position of the threshold of extreme values in disorder-induced multifractals. In particular, the value of the absolute maximum p_{max} will be such that $y = \ln p_{max} - \ln p_m^{(typ)}$ is a random variable of the order of unity.

6. Discussion and Conclusion

In conclusion, we have studied, both analytically and numerically, the strongly fluctuating multifractal pattern in structure of those values of the ideal $1/f$ noise which exceed a given high level comparable with the absolute maximum of the signal. The exploitation of the thermodynamic formalism allowed us to translate the distribution of the partition function found in the previous studies[30, 31] to a similar distribution for the counting functions. The latter is then characterized by a power-law forward tail giving rise to a parametric difference between the mean and the typical value of the counting function when the level position approaches threshold x_m of extreme values. This difference, which can be traced back to the logarithmic correlations inherent in the $1/f$ noise explains the universal coefficient of the subleading term in the position of the threshold x_m .

We have also performed a direct numerical simulations of the $1/f$ signal and calculated numerically the lowest two moments of the partition function. This served to demonstrate that for the samples of $M \sim 10^6$ points the numerics follows $M = \infty$ results rather faithfully. Performing the same check for the counting function moments

however showed that truly asymptotic results can be never achieved even with very moderate accuracy due to prohibitively slow convergence. Instead, even for M as big as $M \sim 10^9$ one has still use a more elaborate finite- M formulas which we derived in the present paper for that goal. This lesson may prove important in view of the growing interest in numerical simulations of related systems arising in the framework of the Random Matrix Theory and the Riemann zeta function along the critical line [47].

Finally, by comparing the results obtained for $1/f$ noises in our paper with those known to hold for multifractal patterns of wave-function intensity at the points of Anderson transitions[9, 14] we propose a quite general formula for the position of extreme values in generic disorder-generated multifractal patterns with non-parabolic singularity spectra. We hope that such prediction can be checked against the accurate numerical data in random multifractals of various origin, and will generate further interest in statistics of high and extreme values in such multifractal patterns. We leave this issue as well as a related, but much more difficult question about actual statistics of the counting function in the region of extreme states for future investigations. For $1/f$ noise the latter should involve understanding of how the so-called "freezing phenomena" known to have profound influence on the behaviour of the partition function Z_q with $|q| > 1$ are reflected in the thermodynamic formalism correspondence between Z_q and the counting function. Note that the freezing mechanism suggested in [29, 30, 31] predicts for $|q| > 1$ the tail behaviour for the distribution of $Z = Z_q$ to be $\mathcal{P}_{|q|>1}(Z) \sim Z^{-(1+\frac{1}{q})} \ln Z$. It is based on considering properties of the generating function $g_q(y) = \overline{\exp\{-e^{-qy} Z_q / Z_e(q)\}}$ which in the limit $M \rightarrow \infty$ is conjectured to stay q -independent (i.e. "frozen") to the value $g_{q=1}(y)$ for all $q > q_c = 1$. The factor $\ln Z$ in $\mathcal{P}_{|q|>1}(Z)$ plays a prominent role and is believed to be a universal feature within the class of Gaussian logarithmically correlated fields. Note that such factor will be precisely absent for i.i.d. case. To that end it is curious to observe that in the context of the Anderson Localization [14, 9] a certain use of the thermodynamic formalism for the IPR's combined with a clever heuristic power counting lead to the distribution for $I = I_q$ with $q > q_c = f'(\alpha_-)$ of the form $\mathcal{P}_{q>q_c}(I) \sim I^{-(1+\frac{q_c}{q})}$. It is most natural to suspect that the logarithmic factor $\ln I$ may be present in the $\mathcal{P}_{q>q_c}(I)$ as well, and the accuracy of the procedure used in [14, 9] was simply not enough to account for it. Closely related questions are whether the generating function $\tilde{g}_q(y) = \overline{\exp\{-e^{-qy} I_q / I_q^{(t)}\}}$ will be actually q -independent for $q > q_c = f'(\alpha_-)$ for general non-parabolic multifractals and whether the probability density of the logarithm of the (appropriately shifted) absolute maximum $y = \ln p_m^{(typ)} - \ln p_{max}$, with $p_m^{(typ)}$ given by (40), will show a characteristic non-Gumbel tail $|y|e^{-|y|}$, $y \rightarrow -\infty$ [29, 30, 31] as our extended analogy would suggest. All these intriguing issues certainly deserve further investigation, both numerically and

analytically.

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Appendix A: Variance of the counting function for Gaussian sequences.

Suppose we have a sequence of M correlated Gaussian-distributed variables V_1, \dots, V_M characterized by the common variances $\overline{V_i^2} = c_0$ and covariances $\overline{V_i V_j} = c_{ij}$, $i \neq j$. Define

$$\Delta_M(a) = \sum_{i \neq j} \left[\overline{\theta(V_i - a) \theta(V_j - a)} - \overline{\theta(V_i - a)} \cdot \overline{\theta(V_j - a)} \right] \quad (.1)$$

Our goal is to show the validity of the following expression

$$\Delta_M(a) = \frac{1}{\pi} \exp\left(-\frac{a^2}{2c_0}\right) \sum_{i \neq j}^M \int_{h_{ij}}^1 \frac{d\tau}{1 + \tau^2} e^{-\frac{a^2}{2c_0} \tau^2}, \quad h_{ij} = \sqrt{\frac{c_0 - c_{ij}}{c_0 + c_{ij}}} \quad (.2)$$

For proving it we need the following

Proposition. Suppose V_1, V_2 are two Gaussian-distributed variables characterized by the common variances $\overline{V_1^2} = \overline{V_2^2} = c_0$ and the covariance $\overline{V_1 V_2} = c$. Then for any two functions $f_1(V)$ and $f_2(V)$ with finite means $\overline{f_{1,2}(v)}$ holds the identity

$$\frac{\partial}{\partial c} \left[\overline{f_1(V_1) f_2(V_2)} \right] = \overline{f'_1(V_1) f'_2(V_2)} \quad (.3)$$

where $f' \equiv \frac{df}{dv}$. To verify the proposition we introduce the vector $\mathbf{v} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$, denote

$\hat{C} = \begin{pmatrix} c_0 & c \\ c & c_0 \end{pmatrix}$ and write the joint probability density of V_1 and V_2 as

$$\mathcal{P}(\mathbf{v}) = \frac{e^{-\mathbf{v}^T \hat{C}^{-1} \mathbf{v}}}{2\pi \sqrt{\det \hat{C}}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \phi^T \hat{C} \phi + i(V_1 \phi_1 + V_2 \phi_2)} \frac{d\phi_1 d\phi_2}{2\pi} \quad (.4)$$

from which it is immediately clear that

$$\frac{\partial}{\partial c} \mathcal{P}(\mathbf{v}) = \frac{\partial^2}{\partial V_1 \partial V_2} \mathcal{P}(\mathbf{v}) \quad (.5)$$

This implies:

$$\frac{\partial}{\partial c} \left[\overline{f_1(V) f_2(V)} \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(V_1) f_2(V_2) \frac{\partial^2}{\partial V_1 \partial V_2} \mathcal{P}(\mathbf{v}) dV_1 dV_2 = \overline{f'_1(V) f'_2(V)} \quad (.6)$$

where the last equality follows after integration by parts.

Now applying the proposition to the particular case $f_1(V) = f_2(V) = \theta(V - a)$ gives in view of $\frac{\partial}{\partial V} \theta(V - a) = \delta(V - a)$ the relation

$$\frac{\partial}{\partial c} \left[\overline{\theta(V_1 - a) \theta(V_2 - a)} \right] = \mathcal{P}(V_1 = a, V_2 = a) = \frac{1}{2\pi \sqrt{c_0^2 - c^2}} e^{-\frac{a^2}{c_0 + c}} \quad (.7)$$

and since $\overline{\theta(V_1 - a)} = \overline{\theta(V_2 - a)}$ is independent of c the integration shows that

$$D_{12} = \overline{\theta(V_1 - a) \theta(V_2 - a)} - \overline{\theta(V_1 - a)} \cdot \overline{\theta(V_2 - a)} = \frac{2}{\pi} \int_0^c \frac{du}{\sqrt{c_0^2 - u^2}} e^{-\frac{a^2}{c_0 + u}} \quad (.8)$$

Finally, introducing the variable $\tau = \sqrt{\frac{c_0 - u}{c_0 + u}}$ we convert the above expression to the form

$$D_{12} = \frac{4}{\pi} \exp\left(-\frac{a^2}{2c_0}\right) \int_h^1 \frac{d\tau}{1 + \tau^2} e^{-\frac{a^2}{2c_0} \tau^2}, \quad h = \sqrt{\frac{c_0 - c}{c_0 + c}} \quad (.9)$$

which after applying to each pair of V_i, V_j in (.1) immediately implies (.2).

Appendix B: Large- M asymptotic of the $\delta_M^{(n)}(x)$ for $\sqrt{2} < x < 2$.

Our goal is to extract the large- M asymptotic behaviour of the integral featuring in (.1), that is:

$$J_M(x) = \int_{\frac{\pi}{M}}^{\frac{\pi}{2}} d\theta \int_{h_\theta}^1 \frac{d\tau}{1 + \tau^2} e^{-\ln M x^2 \tau^2 / 4}, \quad h_\theta = \sqrt{\frac{\ln M + \ln |2 \sin \theta|}{\ln M - \ln |2 \sin \theta|}} \quad (.1)$$

For $x > \sqrt{2}$ we anticipate that the main contribution for $M \rightarrow \infty$ comes from $\theta \rightarrow 0$ and rescale the integration variable as $\theta = \pi M^{-u}, u \in (0, 1)$. Such a rescaling allows us to write $h(\theta) \rightarrow h(u) = \sqrt{\frac{1-u}{1+u}}$ so that we have

$$\begin{aligned} J_{M \gg 1}(x) &\approx -\pi \int_0^1 d(e^{-\ln M u}) \int_{h(u)}^1 \frac{d\tau}{1 + \tau^2} e^{-\ln M x^2 \tau^2 / 4} \\ &= \pi M^{-1} \int_0^1 \frac{d\tau}{1 + \tau^2} e^{-\ln M x^2 \tau^2 / 4} - \frac{\pi}{2} \int_0^1 du h'(u) (1 + u) e^{-\ln M \left(u + \frac{x^2}{4} \frac{1-u}{1+u}\right)} \end{aligned} \quad (.2)$$

For $\ln M \gg 1$ the first integral is dominated by the lower limit and yields the leading order contribution $J_{M \gg 1}^{(I)}(x) \approx \frac{\pi}{Mx} \sqrt{\frac{2\pi}{\ln M}}$. Second integral is dominated by

the vicinity of the minimum of the function $\mathcal{L}(u) = u + \frac{x^2}{4} \frac{1-u}{1+u}$ achieved at $u_* = \frac{x}{\sqrt{2}} - 1$. For $\sqrt{2} < x < 2\sqrt{2}$ we have $u_* \in (0, 1)$ so applying the Laplace method. Using $\mathcal{L}(u_*) = -1 + \sqrt{2}x - x^2/4$ and $\frac{d^2}{du^2}\mathcal{L}(u = u_*) = 2\sqrt{2}/x$ we find the leading order contribution given by:

$$J_{M \gg 1}^{(II)}(x) \approx \sqrt{\frac{\pi^3}{2\sqrt{2} \ln M (2\sqrt{2} - x)}} M^{1 - \sqrt{2}x + \frac{x^2}{4}} \quad (.3)$$

Finally, using asymptotic formula $\text{Erfc}\left(\frac{x}{2}\sqrt{\ln M}\right) \approx \frac{2M^{-\frac{x^2}{4}}}{x\sqrt{\pi \ln M}}$ we arrive at the expression (.1).

References

- [1] Paladin G., Vulpiani A. Anomalous Scaling Laws in Multifractal Objects, *Phys. Rep.* **156** (1987) 147-225
- [2] Stanley H. E. , Meakin P. Multifractal Phenomena in Physics and Chemistry *Nature* **35** (1988), 405-409
- [3] Hubert P., Tessier Y., Lovejoy S., Schertzer D., Schmitt F., Ladoy P. , Carbonnel J.P., Violette S., Desurosne I. Multifractals and extreme rainfall events *Geophys. Res. Lett.*, **20** (1993), 931-934
- [4] Bouchaud J.-P., Potters M., Meyer M. Apparent multifractality in financial time series *Eur. Phys. J. B* **13** (2000) 595-599
- [5] Bacry E., , Delour J., Muzy J.F. Modelling financial time series using multifractal random walks, *Physica A* **299** (2001) 84-92
- [6] Meneveau C., Sreenivasan KR The multifractal Nature of turbulent Energy-dissipation 1991 J. Fluid Mech **224** 429-484
- [7] Schertzer D., Lovejoy S., Schmitt F., Chigirinskaya Y., Marsan D. Multifractal Cascade Dynamics and Turbulent Intermittency *Fractals* **5** (1997) 427-471
- [8] Halsey T.C., Honda K., and Duplantier B. Multifractal Dimensions for Branched Growth, *J. Stat. Phys.* **85** (1996) 681-743
- [9] Evers F. and Mirlin A.D. Anderson Transitions *Rev. Mod. Phys.* **80** (2008) 1355-1417
- [10] Duplantier B., Ludwig A.W.W. Multifractals, Operator Product Expansions, and Field Theory, *Phys. Rev. Lett.* **66** (1991) 247-251
- [11] Meneveau C., Sreenivasan K.R. Measurement of $f(\alpha)$ from scaling of histograms, and applications to Dynamical Systems and fully developed turbulence, *Physics Letters A* **137** (1989) 103-112
- [12] Touchette H. and Beck C. Nonconcave entropies in multifractals and the thermodynamic formalism *J. Stat. Phys.* **125** (2006) 459-475
- [13] Foster M.S., Ryu S., and Ludwig A.W.W. Termination of typical wave-function multifractal spectra at the Anderson metal-insulator transition: Field theory description using the functional renormalization group *Phys. Rev. B* **80** (2009) 075101 [21pp]
- [14] Mirlin A.D. , Evers F. Multifractality and critical fluctuations at the Anderson transition *Phys. Rev. B* **62** (2000) 7920-7933.
- [15] Bogomolny E. Giraud O. Perturbation approach to multifractal dimensions for certain critical random matrix ensembles. *Phys. Rev. E* **84** (2011) 036212 [12pp]
- [16] Duplantier B. Conformally Invariant Fractals and Potential Theory *Phys.Rev.Lett.* **84** (2000) 1363-1367

- [17] Rushkin I., Bettelheim E., Gruzberg I.A. , Wiegmann P. Critical curves in conformally invariant statistical systems. *J. Phys. A: Math. Theor.***40** (2006) 2165-2203
- [18] Castillo H. E.; Chamon C. C.; Fradkin E.; Goldbart P. M. and Mudry C. Exact calculation of multifractal exponents of the critical wave function of Dirac fermions in a random magnetic field. *Phys. Rev. B* **56** (1997) 10668-10677
- [19] Monthus C and Garel T Random cascade models of multifractality: real-space renormalization and travelling waves *J Stat Mech: Th. Exp.* **2010** P06014 [12pp]
- [20] Fyodorov Y.V. Pre-freezing of multifractal exponents in random energy models with a logarithmically correlated potential *J. Stat. Mech.* **2009** P07022 [12pp]
- [21] Mandelbrot B. Negative Fractal Dimensions and Multifractals *Physica A* **163** (1990), 306-315; Chabra A.B. and Sreenivasan K.R. *Phys. Rev. A* **43** (1991) 1114-1117
- [22] Fyodorov Y.V. Multifractality and freezing phenomena in random energy landscapes: An introduction *Physica A* **389** (2010) 4229-4259
- [23] Milotti E. A pedagogical review of $1/f$ noise. (2002) e-preprint Arxiv: physics/0204033
- [24] Pelligrini, B.; Saletti, R. ; Terreni, P., Prudenziati, M. $1/f$ noise in thick-film resistors as an effect of tunnel and thermally activated emissions, from measures versus frequency and temperature. *Phys. Rev. B* **27** (1983) 1233-1243
- [25] Pal A.N., Bol A.A., and Ghosh A. Large low-frequency resistance noise in chemical vapor deposited graphene. *Appl. Phys. Lett.* **97** (2010) 133504 [3pp]
- [26] Koverda V.P., Skokov V.N., and Skripov V.P. $1/f$ Noise in a nonequilibrium phase transition: Experiment and mathematical model. *JETP* **86**(1998) 953-958
- [27] Allegrini P., Menicucci D., Bedini, R., Fronzoni L., Gemignani A., Grigolini P., West B.J., and Paradisi P. Spontaneous brain activity as a source of ideal $1/f$ noise. *Phys. Rev. E* **80**(2009) 061914 [13pp]
- [28] Hooze F.N., Bobberet P.A. On the correlation function of $1/f$ noise. *Physica B* **239** (1997) 223-230
- [29] Carpentier D. and Le Doussal P. Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in Liouville and sinh-Gordon models. *Phys. Rev. E* **63** (2001) 026110 [33pp]
- [30] Fyodorov Y. V. and Bouchaud J. P. Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential *J. Phys. A: Math. Theor.* **41** (2008) 372001 (12pp)
- [31] Fyodorov Y.V., Le Doussal P., and Rosso A. Statistical mechanics of logarithmic REM: duality, freezing and extreme value statistics of $1/f$ noises generated by Gaussian free fields. *J. Stat. Mech.*, 2009, P10005 (31pp)
- [32] Fyodorov Y.V., Le Doussal P., and Rosso A. Freezing Transition in Decaying Burgers Turbulence and Random Matrix Dualities. *Europhys. Lett.* **90** (2010) 60004 (6 pages)
- [33] Relano A., , Gomez, J. M. G., Molina, R.A., and Retamosa J. Quantum Chaos and $1/f$ Noise. *Phys. Rev. Lett.* **89**, (2002) 244102 [4pp]
- [34] Bacry E., Kozhemyak A., Muzy J.F. Continuous cascade models for asset returns. *J. Econ. Dyn. Control.* **32** (2008) 156-199
- [35] Saakian D.B., Martirosyan A., Hu C.-K., and Struzik Z.R. Exact probability distribution function for multifractal random walk models of stocks *EPL* **95** (2011) 28007 [5pp]
- [36] Mordant N., Delour J., Leveque E., Arneodo A., Pinton J.F. Long time correlations in Lagrangian dynamics: a key to intermittency in turbulence. *Phys. Rev. Lett.* **89** (2002) 254502 [4pp]
- [37] Muzy J-F., Baile B., Poggi P. Intermittency of surface-layer wind velocity series in the mesoscale range *Phys. Rev. E* **81** (2010) 056308 [12pp]
- [38] Baile, R., Muzy J-F. Spatial Intermittency of Surface Layer Wind Fluctuations at Mesoscopic Range. *Phys.Rev.Lett.* **105** (2011) 254501 [4pp]

- [39] Bacry E., Muzy, J.F. Log-infinitely divisible multifractal processes. *Commun. Math. Phys.* **236** (2003) 449-475
- [40] Ostrovsky D. Mellin Transform of the Limit Lognormal Distribution. *Comm. Math. Phys.* **288** (2009) 287-310
- [41] Ostrovsky D. On the stochastic dependence structure of the Limit Lognormal Process *Rev. Math. Phys.* **23** (2011) 127-154
- [42] Astala, K.; Jones, P. ; Kupiainen, A. ; Saksman, E. Random curves by conformal welding. *Comptes Rendus Mathematique* **348** (2010) 257-262 and Random Conformal Weldings, *Acta Math* **207** (2011) 203-254
- [43] Bolthausen E., Deuschel J.-D., and Giacomin G. Entropic repulsion and the maximum of the two-dimensional harmonic cristal *Ann. Probab.* **29** (2001) 1670-1692
- [44] Daviaud O. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.* **34** (2006) 962-986.
- [45] Arguin L.-P. and Zindy O. Poisson-Dirichlet statistics for the extremes of a log-correlated Gaussian field (2012) e-preprint arXiv:1203.4216
- [46] Duplantier B., Rhodes R., Sheffield S., Vargas S. Critical Gaussian Multiplicative Chaos: Convergence of the Derivative Martingale, (2012) e-preprint arXiv:1206.1671
- [47] Fyodorov Y.V., Hiary G.A., and Keating, J.P. Freezing Transition, Characteristic Polynomials of Random Matrices, and the Riemann Zeta-Function, *Phys. Rev. Lett.* **108** (2012) 170601 [5pp]
- [48] Antal T., Droz M., Györgyi G., and Racz Z. $1/f$ noise and Extreme Value Statistics. *Phys. Rev. Lett.* **87**(2001) 240601 [4pp]
- [49] Ishioka S., Gingl Z., Choic D., Fuchikami N., Amplitude truncation of Gaussian $1/f^a$ noises, *Physics Letters A* **269**, (2000) 712
- [50] Fyodorov Y.V., Khoruzhenko B.A., N. Simm *in progress*
- [51] Derrida B. and Spohn H. Polymers on disordered Trees, Spin Glasses, and Travelling Waves. *J. Stat. Phys.* **51** (1988) 817-840
- [52] Bramson M, Zeitouni O Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field *Comm. Pure Appl. Math.* **65** (2012) 1-20
- [53] Derrida B. Random Energy Model: an Exactly Solvable model of disordered systems. *Phys. Rev. B* **24** (1981) 2613-2626